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Comparing instanton contributions with exact results in $N=2$ supersymmetric scale invariant theories

Philip C. Argyres[ab] **and Sophie Pelland**[a]

[a]*Newman Laboratory, Cornell University*
Ithaca NY 14853
[b]*Institute for Theoretical Physics, University of California*
Santa Barbara CA 93106
E-mail: `argyres@mail.lns.cornell.edu`, `sp74@cornell.edu`

Abstract: We discuss the general issues and ambiguities involved in matching the exact results for the low energy effective action of scale invariant $\mathcal{N}=2$ supersymmetric QCD to those obtained by instanton methods. We resolve the reported disagreements and verify agreement between an infinite series of coefficients in the low energy effective actions calculated in the two approaches. In particular, we show that the exact low-energy effective couplings for SU(N) for all N with $2N$ fundamental hypermultiplets agree at a special vacuum on the Coulomb branch where a large unbroken discrete global symmetry makes the matching of parameters relatively straightforward.

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Contents

1. Introduction

Following [1, 2], the exact low energy effective actions on the Coulomb branches of many $\mathcal{N}=2$ gauge theories have been deduced. The basic method used is typically a kind of analytic continuation, where the constraints of rigid special Kähler geometry, matching onto various weak coupling limits, and matching onto the strongly coupled SU(2) Yang-Mills solution of [1], combine to determine the exact non-perturbative contributions to the low energy effective action for ranges of parameters where the microscopic theory is strongly coupled.

Alternatively, one may calculate these non-perturbative contributions directly using semi-classical instanton techniques. The expansion in instanton number provides a power series expansion of the couplings of the low energy effective theory around weak coupling which should match that of the exact solution. Such a program has been carried out for scale-invariant $\mathrm{SU}(N)$ theories in [3]–[8] where disagreements with the exact solutions were reported.

In this paper we will resolve these conflicts, and perform an infinite series of checks relating the two approaches. We note that the purpose of this paper is to show how to systematically compare the predictions of any two proposed solutions of a given $\mathcal{N}=2$ theory, but not to discuss how to derive a given solution. In particular, an $\mathcal{N}=2$ QCD solution must satisfy an intricate web of consistency requirements following from the interrelations of the given theory with other theories with different numbers of colors or flavors; this paper is silent about these consistency requirements, effectively assuming that the solutions under examination have already met them.

The paper is organized as follows. In the next section we discuss the general issues involved in comparing the exact solutions to the results of instanton calculations. The central point is the freedom to make non-perturbative redefinitions of the parameters and vacuum expectation values that enter into the low energy effective action. This freedom was pointed out in [9, 10] in comparing different expressions for the effective actions of the scale-invariant $\mathcal{N}=2$ SU(3) theory, and has been used in discussions comparing the semi-classical and exact solutions, resolving the above-mentioned disagreements for $\mathcal{N}=2$ SU(2) QCD, and partially resolving the disagreements for $\mathcal{N}=2$ SU(N) QCD [3, 5, 6, 11, 12]. We extend and clarify these treatments by making a systematic analysis of the full set of allowed non-perturbative redefinitions.

In section 3 we illustrate the use of this freedom in the $\mathcal{N}=4$ SU(2) model broken to $\mathcal{N}=2$ by an adjoint hypermultiplet mass, to resolve the "mismatch" reported in [7]. In this case there are, however, no non-trivial checks of the exact result using only the one-instanton result.

In section 4 we turn to the scale invariant $\mathcal{N}=2$ supersymmetric SU(N) theories with $2N$ fundamental flavors. We first apply our general matching requirements to show the equivalence of the various exact solutions for SU(N) with $2N$ flavors proposed in [13, 14, 15], about which there seems to be some doubt in the literature.

Then in section 5 we resolve the conflicts reported in [8] for these theories. In this case there are in principle many non-trivial checks that can be performed for each N even at the one-instanton level; we verify an infinite sub-series of these checks. In particular we show that the one-instanton results and exact methods agree for all N at a special point in the moduli space of vacua.

2. Comparing curves and instantons

In comparing two solutions for the low energy effective actions of any theory, it is necessary to identify the most general map between the parameters and vacuum expectation values (vevs) entering into the two solutions consistent with dimensional analysis, global symmetries, and the meaning of parameters *versus* vevs in the low energy theory. In the supersymmetric case this matching of parameters is also constrained by various supersymmetric selection rules which in the $\mathcal{N}=2$ cases discussed here imply the holomorphic dependence of certain low-energy effective couplings on parameters in the theory.

The low energy effective action on the Coulomb branch of a scale-invariant $\mathcal{N}=2$ SU(N) theory — the specific focus of this paper — depends on a complex gauge coupling constant q, complex bare mass parameters m, and complex vevs u_k. Let us start by reviewing the definitions of these variables.

Gauge coupling. In scale-invariant theories the complex gauge coupling $\tau = (\theta/2\pi) + i(4\pi/g^2)$ is an exactly marginal parameter of the microscopic (high-energy) theory. Because of the angularity of the theta angle (invariance of the theory under integer shifts of τ) it is convenient to parameterize the theory by $q \equiv e^{2\pi i\tau}$, for which weak coupling corresponds to $q=0$.

Masses. Other parameters entering into the microscopic lagrangian are the complex bare masses of hypermultiplet fields. In terms of $\mathcal{N}=1$ superfields, a hypermultiplet in the fundamental representation of SU(N) can be described by two chiral multiplets, Q and $\widetilde{Q}$ transforming in the $\mathbf{N}$ and $\overline{\mathbf{N}}$, respectively. The complex mass parameter m enters as the superpotential coupling

$$\mathcal{L} \supset m \int d^2\theta \, \mathrm{tr}\, Q\widetilde{Q} + \text{c.c.} \tag{2.1}$$

Vevs. Unlike q and m, vevs are not parameters in the microscopic lagrangian, but instead are coordinates on the moduli space of vacua of the low-energy effective action. On the Coulomb branch, the gauge group is generically broken to $\mathrm{U}(1)^{N-1}$, and the vacua form an $N-1$ complex-dimensional space coordinatized by the gauge invariant combinations of the vevs v_a, $a=1,\ldots,N$, of the adjoint scalar field component v of the $\mathcal{N}=2$ vector superfield V. Classically (at weak coupling), writing v as a traceless complex $N\times N$ matrix in the gauge indices, the flat directions have (up to gauge rotations) diagonal vevs with entries v_a

$$\langle v\rangle = \mathrm{diag}\{v_1,\ldots,v_N\}, \tag{2.2}$$

satisfying the tracelessness condition $\sum_{a=1}^N v_a = 0$. A subgroup of the gauge group acts by permutations on the v_a, so a basis of gauge invariant coordinates (at least near weak coupling) on the Coulomb branch can be taken to be the symmetric polynomials

$$u_k = (-)^k \sum_{a_1<\cdots<a_{k+1}} v_{a_1}\cdots v_{a_{k+1}}, \tag{2.3}$$

for $k=1,\ldots,N-1$. Note that by the tracelessness condition $u_0\equiv 0$. (Indices from the beginning of the alphabet (a, b, c, d) run from 1 to N, while those from the middle (j, k, ℓ, m) will run from 1 to $N-1$ throughout this paper.)

The above definitions only pin down what is meant by q, m, and u_k at weak coupling, since they only refer to how these parameters appear in the classical (weak coupling) lagrangian, or the relation of the u_k's to vevs of a field computed by the

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classical Higgs mechanism. The essential point to bear in mind in matching to the exact solutions found by the "analytic continuation" methods pioneered in [1, 2] is that the above weak-coupling properties of the parameters and vevs are all that is used to define them. Thus any two quantities, say vevs u_k and $\widetilde{u}_k$, that at weak coupling satisfy the defining relation (2.3) but differ by higher-order perturbative or non-perturbative contributions, are equally valid candidates to be "the" vevs entering in the exact solution. In particular, the parameters and vevs of the exact solution can and therefore generically will differ by such higher-order contributions from the parameters and vevs defined in any specific calculational scheme (such as the $\overline{\mathrm{DR}}$ scheme [16] used in supersymmetric instanton computations [17, 18]).

There are, of course, some restrictions on this "anything that can happen, will" principle. First, the parameters and vevs must satisfy selection rules arising from global symmetries (perhaps explicitly or spontaneously broken) of the theory. For example, there is a global $\mathrm{U}(1)_R$ symmetry in the scale invariant models that we are considering in this paper, under which the v_a's and m all have the same charge. Thus the u_k have charge $k+1$ under this spontaneously broken symmetry, and any redefinition of the u_k's must preserve these charge assignments. Also, the selection rules arising from the unbroken global supersymmetry imply that q, m, and u_k all must enter the low-energy effective action holomorphically, and so only holomorphic redefinitions of these parameters are allowed. A perhaps less obvious restriction arises from the distinction between parameters and vevs in field theories: since for a given set of parameters there is a whole moduli space of vacua, one cannot modify the definition of a parameter (q or m) by making it dependent on the vevs (u_k), though it is allowed to do the reverse.

Even with these restrictions there is a large family of allowed redefinitions of the parameters and vevs of the theory. We can be fairly explicit about the form of this family. Consider two descriptions of the theory, one with parameters q, m, and vevs u_k, and the other with $\widetilde{q}$, $\widetilde{m}$, and $\widetilde{u}_k$.

Couplings. As $\widetilde{q}$ is a parameter it can only depend on q and m, but since it is dimensionless it cannot depend on m. By supersymmetry $\widetilde{q}$ must depend on q holomorphically and by its weak coupling definition, must be asymptotically proportional to q as $q \to 0$. The general relation between $\widetilde{q}$ and q is then

$$\widetilde{q} = c_0 q + c_1 q^2 + c_2 q^3 + \cdots \tag{2.4}$$

for some complex numbers c_n. *Thus, what is meant by the complex coupling in the exact low-energy effective actions of $\mathcal{N} = 2$ gauge theories is only determined up to the infinite series of constants c_n.* The coefficient c_0 has the form of a one-loop threshold contribution (a shift of $1/g^2$), while the c_n for $n \geq 1$ correspond to non-perturbative (instanton) modifications of the coupling.

There are two further points worth remarking on in connection with (2.4). The first has to do with cases where there are more than one mass parameter, say m_1 and m_2. One can then form a holomorphic dimensionless ratio of these masses, m_1/m_2, which one might think will enter into the matching (2.4). However, this cannot happen since then there would be a limit in the space of theories where the weak-coupling equivalence of q and $\widetilde{q}$ would fail. An example to make the point clear is $\widetilde{q} = q + (m_1/m_2)q^2$, which for any fixed m_1/m_2 is an allowed redefinition of the coupling, but fails in the corner of parameter space where $m_1/m_2 \to \infty$ as q^{-2} since it violates the weak-coupling $\widetilde{q} \sim q$ requirement.

The second point has to do with global identifications, or S-dualities, on the space of couplings. The redefinition (2.4) need not preserve any "nice" S-duality action on the couplings, for though covariance under S-duality transformations might be convenient, it is not a physical requirement for any description of a theory; the physically meaningful notions are the topology and complex structure of the space of couplings which are invariant under analytic reparametrizations [19].

Masses. The parameter $\widetilde{m}$ can only depend holomorphically on q and m, and by dimensional considerations and agreement with its weak-coupling definition we have

$$\widetilde{m} = (1 + b_1 q + b_2 q^2 + \cdots)m. \tag{2.5}$$

If there is more than one mass parameter, then they may mix at higher orders in q consistent with any global flavor symmetries.

Vevs. The $\widetilde{u}_k$ can depend on u_k, q, and m holomorphically, and should preserve $\mathrm{U}(1)_R$ charge assignments and their weak-coupling definitions:

$$\begin{aligned}
\widetilde{u}_1 &= \left(1 + a_1^{(1;1)} q + \cdots\right) u_1 + \left(a_1^{(1;0,0)} q + a_2^{(1;0,0)} q^2 + \cdots\right) m^2 \\
\widetilde{u}_2 &= \left(1 + a_1^{(2;2)} q + \cdots\right) u_2 + \left(a_1^{(2;1,0)} q + a_2^{(2;1,0)} q^2 + \cdots\right) u_1 m + \\
&\quad + \left(a_1^{(2;0,0,0)} q + a_2^{(2;0,0,0)} q^2 + \cdots\right) m^3 \\
\widetilde{u}_3 &= \left(1 + a_1^{(3;3)} q + \cdots\right) u_3 + \left(a_1^{(3;2,0)} q + a_2^{(3;2,0)} q^2 + \cdots\right) u_2 m \\
&\quad + \left(a_1^{(3;1,1)} q + a_2^{(3;1,1)} q^2 + \cdots\right) u_1 u_1 + \cdots \\
&\;\;\vdots \\
\widetilde{u}_{N-1} &= \left(1 + a_1^{(N-1;N-1)} q + \cdots\right) u_{N-1} + \cdots + \left(a_1^{(N-1;0,\cdots,0)} q + \cdots\right) m^N ,
\end{aligned} \tag{2.6}$$

where the $a_n^{(k;\{k_r\})}$ are arbitrary complex numbers. Recall that m has $\mathrm{U}(1)_R$ charge 1 while the u_k have charges $k+1$. Thus the condition that the general term in $\widetilde{u}_k$,

$$a_n^{(k;\{k_r\})} q^n \prod_r u_{k_r} \tag{2.7}$$

(where we have defined $u_0 \equiv m$), have the correct $\mathrm{U}(1)_R$ charge is $\sum_r (k_r+1) = k+1$. Note that for similar reasons that ratios of masses could not appear in (2.4), ratios of masses or vevs do not appear in (2.6).

Finally, we can remove a small part of this ambiguity by dimensional analysis. So far we have not specified any scale in either of our two descriptions of the theory, so we have the freedom to equate, by *fiat*, any pair of variables of the same dimension (and consistent with the symmetries) to define how scales in one description of the theory are related to scales in the other. In section 5 we will find it convenient to use this freedom to set $\widetilde{u}_{N-1} = u_{N-1}$ in the last equation of (2.6) — i.e., set $a_n^{(N-1;\cdots)} = 0$.

The general matching formulas (2.4)–(2.6) are the main result of this section. Their coefficients represent an inherent ambiguity in any prediction derived from the exact low-energy effective action of $\mathcal{N} = 2$ supersymmetric gauge theory. In the next two sections we will describe two very simple examples of how the use of these matching relations resolves some apparent contradictions or ambiguities reported in the literature.

3. Comparing curves and instantons in SU(2) with massive adjoint matter

In the one-instanton analysis of SU(2) with a massive adjoint hypermultiplet given in [7], a mismatch with the exact curve of [2] is reported. The authors of [7] determine the one instanton correction to the position in the u_1-plane where a component of the hypermultiplet becomes massless. (Here $u_1 = u$ in the notation of [7].) They find the singularity occurs at

$$u_1 = \frac{1}{4}m^2 + \frac{3}{2}q\, m^2 + \mathcal{O}(q^2)m^2\,, \tag{3.1}$$

while the curve of [2] gives

$$\widetilde{u}_1 = \frac{1}{4}\widetilde{m}^2 + 6\widetilde{q}\,\widetilde{m}^2 + \mathcal{O}(\widetilde{q}^2)\widetilde{m}^2\,. \tag{3.2}$$

Here we have designated all the curve quantities by tildes, to differentiate them from those entering the semi-classical analysis. (Note that in (3.2) $\widetilde{u}_1 = u$ in the notation of [2], and is *not* the $\widetilde{u}$ variable of that paper!)

The resolution of this discrepancy is obvious in light of our matching relations (2.4)–(2.6): what is being computed is the relation between the $\widetilde{u}_1$ variable used in the exact solution to the u_1 variable used in the instanton analysis. Given that to leading (semi-classical) order the authors of [7] matched q, m^2, and u_1 to $\widetilde{q}$, $\widetilde{m}^2$, and $\widetilde{u}_1$, so that

$$\begin{aligned}
\widetilde{u}_1 &= u_1 + a_1^{(1;1)} q\, u_1 + a_1^{(1;0,0)} q\, m^2 + \mathcal{O}(q^2 u_1, q^2 m^2)\\
\widetilde{m}^2 &= m^2 + b_1 q\, m^2 + \mathcal{O}(q^2 m^2)\\
\widetilde{q} &= q + \mathcal{O}(q^2)\,,
\end{aligned} \tag{3.3}$$

for some complex numbers $a_n^{(1;k_r)}$ and b_n, the relations (3.1) and (3.2) imply only that

$$a_1^{(1;1)} + 4a_1^{(1;0,0)} - b_1 = 18\,. \tag{3.4}$$

This gives a restriction on the change of variables needed to match the curve to the instanton results, but does not affect any physical predictions. One needs to compute two-instanton corrections in this theory before one can make a physical check of the two approaches, as noted in [7].

4. Comparing different curves for $\mathrm{SU}(N)$ with $2N$ flavors

The main computation of this paper, in section 5, will be an example checking that the physical predictions (after fixing the relations between variables) of the exact curve and of the one-instanton computations match. The theories for which we do this are the scale-invariant $\mathrm{SU}(N)$ theories with $2N$ flavors of massless fundamental hypermultiplets.

We will use the exact curve in the form

$$y^2 = \left(x^N - \sum_{k=1}^{N-1} \widetilde{u}_k x^{N-k-1}\right)^2 - \widetilde{q}\prod_{r=1}^{2N}\left(x + \frac{\widetilde{m}_r}{\sqrt{2}}\right). \tag{4.1}$$

As this form of the curve does not quite correspond to any of the forms [13, 14, 15] appearing in the literature, and also since some uncertainty about the equivalence of these curves has been expressed [8], in this section we will show that all these forms are equivalent to (4.1).

We start with the form of the curve derived in [13]:

$$y^2 = \left(x^N - \sum_{k=1}^{N-1} u_k x^{N-k-1}\right)^2 - (1-g^2)\prod_{r=1}^{2N}(x + (g-1)\mu + m_r)\,. \tag{4.2}$$

Here $\mu \equiv (1/2N)\sum_r m_r$ and g is some function of the coupling which has the expansion

$$g(q) = 1 - 2q + \mathcal{O}(q^2) \tag{4.3}$$

near weak coupling. It is clear that defining

$$\begin{aligned}
\widetilde{q} &= 1 - g^2 = 4q + \mathcal{O}(q^2)\\
\frac{\widetilde{m}_r}{\sqrt{2}} &= m_r + (g-1)\mu = m_r - \frac{q}{N}\sum_s m_s + \mathcal{O}(q^2 m_s)\\
\widetilde{u}_k &= u_k\,,
\end{aligned} \tag{4.4}$$

makes (4.2) and (4.1) equivalent. Since these relations are of the form of the allowed redefinitions of variables we derived in section 2, this then shows the physical equivalence of these curves.[1]

We now turn to the form of the curve described in [14]:

$$\widetilde{y}^2 = \left(\widetilde{x}^N - \ell\sum_{k=1}^{N-1} u_k\widetilde{x}^{N-k-1} + \frac{1}{4}L\sum_{s=0}^{N} t_s\widetilde{x}^{N-s}\right)^2 - L\prod_{r=1}^{2N}(\widetilde{x}+\ell m_r)\,, \tag{4.5}$$

where

$$\begin{aligned} L(q) &= 64q + \mathcal{O}(q^2) \\ \ell(q) &= 1 + \mathcal{O}(q) \\ t_s(m_r) &= \sum_{r_1>\cdots>r_s} m_{r_1}\cdots m_{r_s}; \qquad t_0 \equiv 1\,. \end{aligned} \tag{4.6}$$

If we rescale the u_k by ℓ and shift by $(L/4)t_{k+1}$, rescale the masses by ℓ, and define $\widetilde{q} = L$, (all of which are allowed redefinitions by our previous discussion) then (4.5) becomes of the form (4.1) except for two minor discrepancies: in the squared term on the right hand side there is an $\widetilde{x}^{N-1}$ term with coefficient $(L/4)t_1 \sim qm_r$ and the $\widetilde{x}^N$ term has coefficient $1+(L/4) \sim 1+\mathcal{O}(q)$. These discrepancies can be removed by a shift of the (dummy) $\widetilde{x}$ variable by $\sim Lm_r$ and a rescaling of $\widetilde{y}$ by $1+(L/4)$. These then require shifts and rescalings of the coupling, masses, and vevs, but since the shifts are all $\mathcal{O}(q)$ and the rescalings are $\sim 1+\mathcal{O}(q)$, they are also allowed redefinitions of our variables. Explicitly, the change of variables

$$\begin{aligned} \widetilde{x} &= x + \alpha \\ \widetilde{y} &= \frac{1}{4}(4+L)y \\ \widetilde{q} &= \frac{16L}{(4+L)^2} \sim 64q + \mathcal{O}(q^2) \\ \frac{\widetilde{m}_r}{\sqrt{2}} &= \ell m_r + \alpha \sim m_r + \mathcal{O}(qm_s) \\ \widetilde{u}_k &= k\binom{N}{k+1}\alpha^{k+1} + \sum_{j=1}^{k}\frac{1}{4+L}\binom{N-1-j}{k-j}(4\ell u_j - Lt_{j+1})\alpha^{k-j} \\ &\sim u_k + \mathcal{O}(qm_r, qu_j)\,, \end{aligned} \tag{4.7}$$

where

$$\alpha \equiv -\frac{1}{N}\frac{L}{4+L}t_1 \sim -\frac{16}{N}q\sum_r m_r + \mathcal{O}(q^2 m_s)\,, \tag{4.8}$$

takes (4.5) to (4.1).

[1]The factor of $\sqrt{2}$ difference in the mass parameters reflects an incorrect factor of $\sqrt{2}$ in the definition of hypermultiplet masses in [13].

Finally, the form of the scale-invariant 6 flavor SU(3) curve proposed in [15] is

$$\widetilde{y}^2 = (\xi x^3 - u_1 x - u_2)^2 - (\xi^2 - 1) \prod_{r=1}^{6} \left(x - \frac{1}{6}(1 - \xi^{-1}) \sum_s m_s + m_r \right) , \tag{4.9}$$

where

$$\xi(q) = 1 + \mathcal{O}(q) \,. \tag{4.10}$$

Then, very much as above, if we change variables to

$$\begin{aligned} \widetilde{y} &= \xi y \\ \widetilde{q} &= 1 - \xi^{-2} \\ \frac{\widetilde{m}_r}{\sqrt{2}} &= m_r - \frac{1}{6}(1 - \xi^{-1}) \sum_s m_s \\ \widetilde{u}_k &= \xi^{-1} u_k \,, \end{aligned} \tag{4.11}$$

we recover (4.1).

5. Comparing curves and instantons in SU(N) with $2N$ flavors

We now come to the main calculation of this paper, where we resolve the difficulty, reported in [8], with matching a one-instanton calculation to the exact curve for the scale-invariant $2N$ flavor SU(N) theory.

The authors of [8] proposed to relate the coupling parameter $\widetilde{\tau}$ of the curve (4.1) to the low energy matrix of effective couplings $\widetilde{\tau}^{jk}$ at a special vacuum in the moduli space of the theory, where *classically* $\widetilde{\tau}^{jk} = \widetilde{\tau}\, \tau_{cl}^{jk}$ with

$$\tau_{cl}^{jk} \equiv 1 + \delta^{jk} = \begin{pmatrix} 2 & 1 & 1 & \dots \\ 1 & 2 & 1 & \dots \\ 1 & 1 & 2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} . \tag{5.1}$$

The low-energy matrix of $\mathrm{U}(1)^{N-1}$ couplings are the coefficients of the $\mathrm{U}(1)^{N-1}$ gauge kinetic term,

$$-\frac{1}{16\pi} \operatorname{Im} \tau^{jk} F_j^+ \cdot F_k^+ \,, \tag{5.2}$$

in the low-energy effective action on the Coulomb branch.[2] The problem is then that for SU(N) with $N > 3$ it can be proven [10, 20] that $\widetilde{\tau}^{jk}$ is not proportional to τ_{cl}^{jk} at *any* point on the Coulomb branch.

The first point to make towards resolving this difficulty is to note that $\widetilde{\tau}$ is the microscopic coupling parameter of the theory, and as such should not be expected in general to satisfy any simple relation to the matrix of low-energy couplings $\widetilde{\tau}^{jk}$. The

[2] Here $F_j^+ \equiv F_j^{\mu\nu} - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma} F_{j,\rho\sigma}$ is the self-dual field strength of the jth U(1) factor.

low-energy coupling is some function of the microscopic coupling parameter and the vevs. The instanton methods also calculate these low-energy coupling functions as an expansion in q:

$$\tau^{jk} = \frac{\ln q}{2\pi i}\tau^{jk}_{cl} + \sum_{n=0}^{\infty} q^n \tau^{jk}_n\,, \tag{5.3}$$

where τ^{jk}_0 is the one-loop threshold matching contribution, while τ^{jk}_n for $n > 0$ are the n-instanton contributions; all the τ^{jk}_n are functions of the vevs. The proper way to compare the results of the two methods is to compare their predictions for the matrix of low-energy couplings as functions of the coupling and the vevs modulo the allowed redefinitions (2.4) and (2.6) of the variables. (Note that here we are setting all mass parameters to zero.) There is also the inherent ambiguity in the definition of the low-energy couplings themselves by electric-magnetic duality transformations, which must also be taken into account; we will encounter examples of this below.

To compare the physical predictions of the exact and instanton methods, we must relate the parameter and vevs on the two sides. On the exact side, we use the variables $\widetilde{q}$ and $\widetilde{u}_k$, while on the instanton side, we use q and u_k. (In general, quantities associated with the exact solutions will have tildes, those in the instanton calculations, not.) The technical problem of how to carry out this matching in practice remains, and might seem difficult in light of the large number of undetermined coefficients in the matching relations (2.6). However, we can use the following simple trick to organize the comparison of the two methods: we expand about a special direction in the moduli space where the $\mathrm{U}(1)_R$ symmetry is partly restored to a $\mathbb{Z}_N$ symmetry and makes the matching in (2.6) unambiguous. In particular, along the line $u_1 = u_2 = \cdots = u_{N-2} = 0$ in the Coulomb branch (with $m = 0$), (2.6) implies simply

$$\begin{aligned} \widetilde{u}_k &= u_k = 0\,, \qquad k = 1, \ldots, N-2 \\ \widetilde{u}_{N-1} &= u_{N-1}\,. \end{aligned} \tag{5.4}$$

Here we have used our freedom to set dimensionful scales to equate $\widetilde{u}_{N-1}$ and u_{N-1} exactly, without any q-dependent factor, as discussed at the end of section 2. Indeed, since we will only be calculating dimensionless couplings, without any loss of generality we set

$$\widetilde{u}_{N-1} = u_{N-1} = 1 \tag{5.5}$$

from now on. We will refer to this point on the Coulomb branch as the special vacuum. It turns out to be more convenient to express the vevs on the instanton side in terms of the eigenvalues v_a, $a = 1, \ldots, N$ satisfying the tracelessness condition $\sum_{a=1}^{N} v_a = 0$, and related to the u_k by (2.3). In these variables, the special vacuum is at

$$v_a|_{sv} = \omega^a, \qquad \text{where} \quad \omega \equiv e^{2\pi i/N}\,. \tag{5.6}$$

The leading terms in the matching relation for the couplings are

$$\widetilde{q} = C_0 q + C_1 q^2 + \mathcal{O}(q^3)\,, \tag{5.7}$$

for arbitrary complex constants C_0 and C_1. (The C_0 coefficient is computed by a one-loop threshold effect.)

We now compare the curve and instanton predictions by computing the matrix of low-energy couplings in each approach, expanding about weak coupling at the special vacuum:

$$\begin{aligned}\tau^{jk} &= \frac{\ln q}{2\pi i}\tau_{cl}^{jk} + \tau_0^{jk} + q\tau_1^{jk} + \mathcal{O}(q^2)\,,\\ \widetilde{\tau}^{jk} &= \frac{\ln \widetilde{q}}{2\pi i}\widetilde{\tau}_{cl}^{jk} + \widetilde{\tau}_0^{jk} + \widetilde{q}\widetilde{\tau}_1^{jk} + \mathcal{O}(q^2)\,.\end{aligned} \tag{5.8}$$

Here we have used the fact that the tree-level Higgs mechanism implies the classical couplings $\tau \cdot \tau_{cl}^{jk}$ where τ_{cl}^{jk} is the constant matrix given in (5.1); the off-diagonal elements appear because of the tracelessness condition for the v_a. Now equating the two expansions term by term in q using the matching relation (5.7) implies

$$\begin{aligned}\tau_{cl}^{jk} &= \widetilde{\tau}_{cl}^{jk}\\ \tau_0^{jk} &= \frac{\ln C_0}{2\pi i}\tau_{cl}^{jk} + \widetilde{\tau}_0^{jk}\\ \tau_1^{jk} &= \frac{C_1}{2\pi i C_0}\tau_{cl}^{jk} + C_0\widetilde{\tau}_1^{jk}\,,\end{aligned} \tag{5.9}$$

at the special vacuum. If there exist numbers C_0, and C_1 that make these equations true, then we will have shown that, to this order in the coupling, the two methods agree.[3] The rest of this paper will be concerned with computing the coefficient matrices in (5.9) and solving for the C_0 and C_1 matching constants to show that the two methods do indeed agree.

5.1 Curve computation

We use for the curve describing the auxiliary Riemann surface Σ whose complex structure encodes $\widetilde{\tau}^{jk}$ for the scale-invariant SU(N) with $2N$ massless flavors the one (4.1) discussed in section 4, which we reproduce here (with masses set to zero):

$$y^2 = \left(x^N - \sum_{k=1}^{N-1}\widetilde{u}_k x^{N-1-k}\right)^2 - \widetilde{q}x^{2N}\,. \tag{5.10}$$

The matrix of low-energy couplings, $\widetilde{\tau}^{jk}$, is defined in terms of data on Σ by

$$\widetilde{\tau}^{jk} = (A^{-1})^j_\ell B^{\ell k}, \qquad \text{with} \quad A^\ell_j = \oint_{\alpha_j}\omega^{(\ell)} \quad \text{and} \quad B^{\ell k} = \oint_{\beta^k}\omega^{(\ell)}\,, \tag{5.11}$$

[3] There is a discrete ambiguity due to electric-magnetic duality redefinitions in the low-energy theory, which we have suppressed. This will be discussed below.

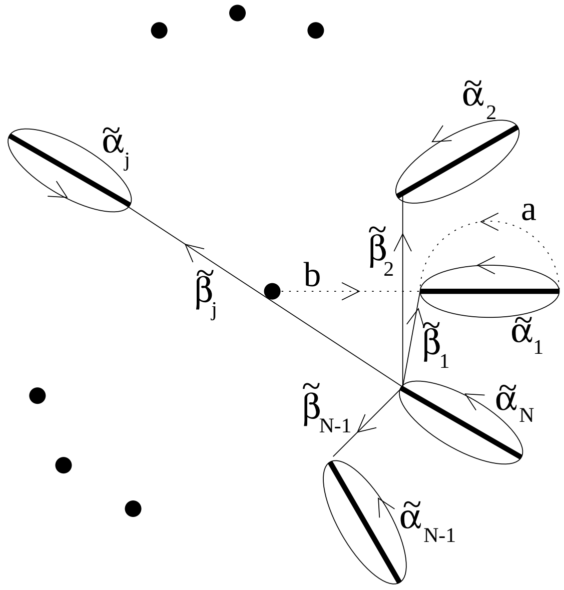

Figure 1: Representation of one sheet of the two-sheeted covering of Σ given in (5.10) with a (non-canonical) basis $\{\widetilde{\alpha}_j, \widetilde{\beta}_k\}$ of cycles in homology. The dark lines represent the N cuts on the x-plane. Only half of the $\widetilde{\beta}_k$ cycles are visible, the other half lying on the second sheet. The dotted lines represent the a and b contours used in appendix A.

where the $\omega^{(\ell)}$ are an arbitrary basis of holomorphic differentials on Σ which can conveniently be taken to be

$$\omega^{(\ell)} = \frac{x^{\ell-1}dx}{y}, \qquad \ell = 1, \ldots, N-1\,, \tag{5.12}$$

and where α_j and β^j form a canonical homology basis on Σ, which is a basis defined to have intersections $\alpha_j \circ \alpha_k = \beta^j \circ \beta^k = 0$, and $\alpha_j \circ \beta^k = \delta_j^k$.

To compute the A_j^ℓ and $B^{\ell k}$ we must first select a canonical homology basis. Figure 1 shows a (non-canonical) homology basis, $\widetilde{\alpha}_j$ and $\widetilde{\beta}^j$, convenient for calculations, in terms of which a canonical homology basis α_j, β^j is:

$$\alpha_j = \widetilde{\alpha}_j\,, \qquad \beta^j = \widetilde{\beta}^j + \sum_{k=1}^{j} \widetilde{\alpha}_k\,. \tag{5.13}$$

A_j^ℓ and $B^{\ell k}$ are evaluated at the special vacuum in appendix A, yielding

$$\begin{aligned}\widetilde{\tau}^{jk} = {} & \frac{\ln \widetilde{q}}{2\pi i}(1+\delta^{jk}) + (1+\delta^{jk})\left(\frac{1}{2} + \frac{i}{\pi}\ln 4N\right) + \frac{i}{\pi}L^{jk} + \\ & + \widetilde{q}\frac{i}{4\pi N^2}\left[(1+\delta^{jk})\frac{1-7N^2}{6} + \frac{2\omega^j}{(1-\omega^j)^2} + \frac{2\omega^k}{(1-\omega^k)^2} - \right. \\ & \left. - (1-\delta^{jk})\frac{2\omega^{j+k}}{(\omega^j-\omega^k)^2}\right],\end{aligned} \tag{5.14}$$

where

$$L^{jk} \equiv (1-\delta^{jk}) \ln \left[\frac{\sin(\pi j/N)\sin(\pi k/N)}{\sin(\pi|j-k|/N)}\right] + \delta^{jk} \ln \left[\sin^2\left(\frac{\pi j}{N}\right)\right]. \tag{5.15}$$

From (5.14) we calculate the right sides of (5.9) to be

$$\frac{\ln C_0}{2\pi i}\tau_{cl}^{jk} + \widetilde{\tau}_0^{jk} = (1+\delta^{jk})\left[\frac{1}{2} + \frac{1}{2\pi i}\ln\left(\frac{C_0}{16N^2}\right)\right] + \frac{i}{\pi}L^{jk}, \tag{5.16}$$

and

$$\begin{aligned} \frac{C_1}{2\pi i C_0}\tau_{cl}^{jk} + C_0\widetilde{\tau}_1^{jk} = \frac{C_0}{4\pi i N^2}\Bigg\{&(1+\delta^{jk})\left[\frac{N^2-1}{6} + \left(1+\frac{2C_1}{C_0^2}\right)N^2\right] - \\ &- \frac{2\omega^j}{(1-\omega^j)^2} - \frac{2\omega^k}{(1-\omega^k)^2} + (1-\delta^{jk})\frac{2\omega^{j+k}}{(\omega^j-\omega^k)^2}\Bigg\}. \end{aligned} \tag{5.17}$$

5.2 The electric-magnetic duality ambiguity

Different choices of canonical homology bases translate into different τ^{jk} related by

$$\tau \to (a\tau+b)(c\tau+d)^{-1}, \tag{5.18}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}(2N-2,\mathbb{Z})$ for $(N-1)\times(N-1)$ integer matrices a, b, c, and d. This is an electric-magnetic duality transformation on the low-energy $\mathrm{U}(1)^{N-1}$ couplings, and is an inherent ambiguity in the meaning of those couplings: any two τ^{jk} related by (5.18) are physically equivalent. In comparing the τ^{jk}'s predicted by the curve and instanton methods, the possibility that they differ by such an $\mathrm{Sp}(2N-2,\mathbb{Z})$ transformation must be taken into account.

This is only a discrete ambiguity, however, since $\mathrm{Sp}(2N-2,\mathbb{Z})$ is a discrete group. In particular, the entries in $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ are integers and therefore q and u_k independent. Thus it suffices to determine this matrix at one vacuum (the special vacuum, in our case).

In the case at hand, we have chosen our canonical basis of cycles $\{\alpha_j, \beta_k\}$ so that the leading (classical) term in the low-energy matches with the semi-classical result: the coefficient of $(\ln q)/(2\pi i)$ in both cases is $\tau_{cl}^{jk} = 1 + \delta^{jk}$. We can ask, given this agreement, what electric-magnetic duality ambiguity remains? Consider two weak-coupling expansions of the low-energy τ_{ij} of the form (5.3) with coefficients τ_n^{jk} and $\widehat{\tau}_n^{jk}$. Then the condition that they be related by an $\mathrm{Sp}(2N-2,\mathbb{Z})$ transformation of the form (5.18) for arbitrary (weak) coupling q implies that $c = 0$ and

$$\begin{aligned} a\tau_{cl} &= \tau_{cl} d \\ a\widehat{\tau}_0 + b &= \tau_0 d \\ a\widehat{\tau}_n &= \tau_n d, \qquad n \geq 1, \end{aligned} \tag{5.19}$$

where we are treating τ and $\widehat{\tau}$ as $(N-1)\times(N-1)$ matrices, and matrix multiplication is understood. The first thing to note is that b, which corresponds to constant shifts of the τ^{jk}, is determined at the one-loop level; we will determine it below. Secondly, the condition that $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{Sp}(2N-2,\mathbb{Z})$ implies that $d=(a^T)^{-1}$, so the only remaining ambiguity lies in the choice of $a \in \mathrm{SL}(N-1,\mathbb{Z})$ which satisfies

$$a\tau_{cl}a^T = \tau_{cl}\,. \tag{5.20}$$

We can determine the possible a satisfying (5.20) as follows. Think of a as a matrix mapping the lattice of integer $(N-1)$-vectors to itself. (5.20) implies that a also preserves the "metric" τ_{cl} on this space. But with this metric, the minimum non-zero length-squared of an integer vector is 2, given by vectors of the form $\pm e_j$ or $e_j - e_k$ where e_j is the vector with a 1 in the jth position and zeros elsewhere. So a must map the basis $\{e_i\}$ to another basis made up of $\pm e_j$ and/or $e_j - e_k$. Up to a permutation matrix, such a matrix must be of the triangular form

$$a_\triangle = \begin{pmatrix} \pm1 & * & * & \cdots & * \\ 0 & \pm1 & * & \cdots & * \\ 0 & 0 & \pm1 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \pm1 \end{pmatrix} \tag{5.21}$$

where the upper triangular entries are either 0 or ±1 with at most one non-zero off-diagonal entry in each column, of opposite sign to the diagonal entry in that column. The general form for a is then

$$a = P_1 a_\triangle P_2, \tag{5.22}$$

for $P_{1,2}$ arbitrary permutation matrices.

It so happens that the matching at the special vacuum using the canonical basis of figure 1, which we perform below, works for $a=1$, so we do not have to exercise the option of more complicated electric-magnetic duality transformations with $a \neq 1$. We will see, however, that we will have to use our freedom to do integer shifts of τ^{jk}, which correspond to electric-magnetic duality transformations with $b \neq 0$.

5.3 One-loop threshold computation

The one-loop correction is a threshold correction, i.e., a renormalization scheme dependent difference between the gauge couplings in the high energy theory and those in the low energy effective theory. Such threshold corrections were calculated in [21] in a dimensional regularization with minimal subtraction scheme, and were converted to the $\overline{\mathrm{DR}}$ scheme in [22], giving the result

$$\begin{aligned} \frac{1}{g_{ab}^2} - \frac{1}{g_0^2} = -\frac{1}{48\pi^2}\Bigg\{ &-21\sum_V \mathrm{Tr}\left(t_a^V t_b^V \ln\frac{M_V}{\mu}\right) + 8\sum_F \mathrm{Tr}\left(t_a^F t_b^F \ln\frac{M_F}{\mu}\right) + \\ &+ \sum_S \mathrm{Tr}\left(t_a^S t_b^S \ln\frac{M_S}{\mu}\right)\Bigg\}. \end{aligned} \tag{5.23}$$

Here μ is an energy scale, g_0^2 is the high energy coupling, $g_{ab}^2(\mu)$ are the low energy couplings, $M_{V,F,S}$ are mass matrices, the t_a's are the properly normalized generators of the low energy unbroken U(1)'s, and the sums are over (real) massive vectors, Dirac fermions, and real scalars. Since a massive $\mathcal{N}=2$ vector multiplet has one massive vector field, one Dirac fermion field, and one real scalar field, all in the adjoint representation, while our $2N$ massive hypermultiplets each have a Dirac fermion and four real scalars, all in the fundamental representation, we get

$$\frac{4\pi i}{g_{ab}^2(\mu)} - \frac{4\pi i}{g_0^2} = \frac{i}{\pi}\left[\mathrm{Tr}\left(t_a^V t_b^V \ln\left|\frac{M_V}{\mu}\right|\right) - 2N\,\mathrm{Tr}\left(t_a^H t_b^H \ln\left|\frac{M_H}{\mu}\right|\right)\right]. \tag{5.24}$$

Here t_a^V, t_a^H are generators of the low energy $\mathrm{U}(1)^{N-1}$ gauge group descended from the Cartan generators in the adjoint and fundamental representations, respectively, of the microscopic SU(N) gauge group. It is easiest to calculate in the case where the high energy theory is U(N) broken down to $\mathrm{U}(1)^N$ and then decouple the extra diagonal U(1) factor at the end. The conventionally normalized generators are then

$$(t_a^V)^{bd}_{ce} = \frac{1}{\sqrt{2}}\delta^b_e\delta^d_c\left(\delta^b_a - \delta^a_c\right), \qquad (t_a^H)^b_c = \frac{1}{\sqrt{2}}\delta^b_c\delta^b_a, \tag{5.25}$$

where $a,b,c,d,e = 1,\ldots,N$ and we are thinking of the N U(1) generators t_a^V as matrices mapping the N^2-dimensional space of adjoint bc indices to the N^2-dimensional space of de indices, and the t_a^H similarly as $N \times N$ matrices acting on the N-dimensional fundamental representation space.

The vector multiplet mass matrix, by the usual Higgs mechanism, is read off from the kinetic term for v, giving

$$\mathrm{Tr}\left([A^\mu, \langle v\rangle]^\dagger [A_\mu, \langle v\rangle]\right) = -\left|v_a - v_b\right|^2 (A^\mu)^a_b\,(A_\mu)^b_a, \tag{5.26}$$

implying that the diagonal A's are massless, and the off-diagonal A's have mass $|v_a - v_b|$, giving the mass matrix

$$|M_V|^{ac}_{bd} = |v_a - v_b|\,\delta^a_d\delta^c_b. \tag{5.27}$$

The masses for the accompanying massive real scalar and Dirac fermion in the $\mathcal{N}=2$ vector multiplet are the same by supersymmetry. The hypermultiplet masses arise from the superpotential term [23]

$$\mathcal{L} \supset \sqrt{2}\int d\theta^2\, \mathrm{tr}(Qv\widetilde{Q}) + \mathrm{c.c.} \tag{5.28}$$

Replacing v by its vev generates a mass term for the hypermultiplets which, comparing to the bare mass term (2.1), implies the hypermultiplet mass matrix

$$|M_H|^a_b = \sqrt{2}\,|v_a|\,\delta^a_b. \tag{5.29}$$

These are the masses for the four real scalars as well as the Dirac fermion in the massive hypermultiplet.

Plugging (5.25), (5.27 and (5.29) into (5.24) gives

$$i\,\mathrm{Im}\,\tau_0^{ab} = \frac{i}{\pi}\left[\delta^{ab}\left(\sum_{c\neq a}^{N}\ln|v_a - v_c| - N\ln|\sqrt{2}v_a|\right) - (1-\delta^{ab})\ln|v_a - v_b|\right], \quad (5.30)$$

where we have used $\tau^{ab} \equiv (\theta^{ab}/2\pi) + i(4\pi/g_{ab}^2)$.

To translate this to the $\mathrm{SU}(N) \to \mathrm{U}(1)^{N-1}$ couplings, we remove the trace piece to make the $(N-1)\times(N-1)$ matrix of low-energy couplings

$$\tau_0^{jk} = \tau^{jk} - \tau^{jN} - \tau^{Nk} + \tau^{NN}, \qquad j,k = 1,\ldots,N-1, \quad (5.31)$$

where τ's on the right refer to the $N\times N$ matrix of couplings computed above in the $\mathrm{U}(N)$ theory. This subtraction just reflects the fact that in the $\mathrm{SU}(N)$ theory, one of the N $\mathrm{U}(1)$ subgroups is not independent of the others since their generators sum to zero. To include the theta angles (making τ complex), we can simply drop the absolute value signs in (5.30). Although there is not a unique way of assigning phases to the arguments of the logarithms, the ambiguity is easily seen to be equivalent to an electric-magnetic duality transformation (an integer shift of the low energy τ^{jk}'s). Thus, the one-loop threshold contribution is

$$\begin{aligned} i\pi\tau_0^{jk} = {} & \delta^{jk}\left(N\ln(\sqrt{2}v_j) - \sum_{a\neq j}^{N}\ln(v_a - v_j)\right) + (1-\delta^{jk})\ln(v_j - v_k) - \\ & - \ln(v_j - v_N) - \ln(v_k - v_N) + N\ln(\sqrt{2}v_N) - \sum_{\ell=1}^{N-1}\ln(v_\ell - v_N). \end{aligned} \quad (5.32)$$

Finally, this expression is not necessarily symmetric under interchange of j and k. But in the low-energy effective action it multiplies a term symmetric on these indices, so we should simply symmetrize (5.32).

At the special vacuum, where $v_a = \omega^a$, (5.32) gives

$$\tau_0^{jk} = (1+\delta^{jk})\left[\frac{1}{2} + \frac{i}{\pi}\ln(2N2^{-N/2})\right] + \frac{i}{\pi}L^{jk}, \quad (5.33)$$

where we have used the fact that $\prod_{k=1}^{N-1}(1-\omega^k) = N$ (see appendix B), have made some integral electric-magnetic duality shifts as in (5.19), and where L^{jk} is given in (5.15).[4] The matching between (5.33) and the one-loop threshold contribution from the curve, given in eq. (5.16) is achieved by choosing

$$C_0 = 2^{N+2}. \quad (5.34)$$

[4]The one-loop contribution to τ^{jk} in the special vacuum was reported in [10] to be $\tau_0^{jk} = (1-\delta^{jk})\frac{1}{2} + (1+\delta^{jk})\frac{i}{\pi}\ln 2N + \frac{i}{\pi}L^{jk}$. After the $\mathrm{Sp}(2N-2,\mathbb{Z})$ electric-magnetic duality shift $\tau_0^{jk} \to \tau_0^{jk} + \delta^{jk}$, this matches our result except for a $(1+\delta^{jk})N\ln 2/(2\pi i)$ term. This is due to the factor of $\sqrt{2}$ in the hypermultiplet masses (5.29), which was neglected in [10].

Note that this value of C_0 differs from the one implicit in the curve found in [13], which gave $C_0 = 4$, whose normalization is expected to satisfy our perturbative matching requirements (in light of the result of [18] of no thresholds in the $\overline{\rm DR}$ scheme). The difference of a factor of $\sqrt{2}^{2N}$ can be traced back to an incorrect extra factor of $\sqrt{2}$ in the superpotential mass term (2.1) of [13].

5.4 One-instanton contribution

The low energy $\mathcal{N} = 2$ effective action is usefully expressed in terms of the prepotential [23], a scalar function of the vector superfields, in terms of which the matrix of low energy couplings is given by

$$\tau^{jk} \equiv \frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial V_j \partial V_k}\,, \tag{5.35}$$

where the V_j are a set of $N-1$ independent vevs on the Coulomb branch. When the superpotential is given as a function of the v_a's, as in the instanton calculation of [8] to which we are comparing the curve results, we can take the $N-1$ independent Coulomb branch vevs to be the first $N-1$ of the v_a used in (2.2). The tracelessness constraint on the v_a then implies that v_N is not an independent variable, and so the low-energy coupling must be computed by

$$\tau^{jk} = \frac{1}{2}\left(\frac{\partial}{\partial v_j} - \frac{\partial}{\partial v_N}\right)\left(\frac{\partial}{\partial v_k} - \frac{\partial}{\partial v_N}\right)\mathcal{F}\,, \tag{5.36}$$

where the partial derivatives are taken treating all N v_a as independent variables. The prepotential expanded about $q = 0$ yields a classical piece, a one-loop correction and instanton corrections:

$$\mathcal{F} = \frac{\ln q}{2\pi i}\left(\sum_{a=1}^{N} v_a^2\right) + \sum_{n=0}^{\infty} q^n \mathcal{F}_n\,. \tag{5.37}$$

Taking derivatives of this expansion as in (5.36) yields the instanton expansion (5.3) of the low energy coupling τ^{jk}.

The one-instanton prepotential for $N_f \leq 2N$ was calculated in [8]. Applying their result to $N_f = 2N$ with massless matter gives the one-instanton prepotential:

$$\mathcal{F}_1 = \frac{C_1' \pi^{2N-1}}{i 2^{2N+2}} \sum_{c \neq d} \frac{(v_c + v_d)^{2N}}{(v_c - v_d)^2} \prod_{b \neq c,d} \frac{1}{(v_b - v_c)(v_b - v_d)}\,, \tag{5.38}$$

where $b, c, d = 1, \ldots, N$ and C_1' is a scheme-dependent constant. Differentiating this expression twice as in (5.36) and evaluating the resulting sums and products in the special vacuum, we obtain after a lengthy computation (see appendix B) the one-instanton contribution to τ^{jk} in the special vacuum:

$$\tau_1^{jk} = \frac{C_1'\pi^{2N}}{4\pi i N^2}\left\{(1+\delta^{jk})\left[\frac{N^2-1}{6} - \binom{2N}{N+1}\frac{N^2}{2^{2N}}\right] - \right.$$
$$\left. - \frac{2\omega^j}{(1-\omega^j)^2} - \frac{2\omega^k}{(1-\omega^k)^2} + (1-\delta^{jk})\frac{2\omega^{j+k}}{(\omega^j-\omega^k)^2}\right\}. \tag{5.39}$$

Comparing this with the one-instanton component of the exact result, given in (5.17), implies

$$\begin{aligned} C_1' &= 2^{N+2}\pi^{-2N} \\ C_1 &= -8\left[2^{2N} + \binom{2N}{N+1}\right]. \end{aligned} \tag{5.40}$$

Thus, we have successfully matched the curve $\widetilde{\tau}^{jk}$ with the semiclassical τ^{jk} in the special vacuum. This determines the first two coefficients C_0 and C_1 in the expansion of $\widetilde{q}$ in terms of q of (5.7). Note that this is a non-trivial matching since the matching of the $N \times N$ matrix of low-energy effective couplings forms a system of $\frac{1}{2}N(N+1)$ equations in 2 unknowns for each N. The existence of a solution for C_0 and C_1 for all N thus provides an infinite number of physical checks matching the predictions of the exact results and the semi-classical results at the special vacuum.

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A. Details of curve computation of $\widetilde{\tau}^{jk}$

Using the $\mathbb{Z}_N$ symmetry of the special vacuum, the computation of A_j^ℓ and $B^{\ell k}$ can be reduced to integrals over the paths a and b shown in figure 1. In particular, defining

$$a_\ell \equiv \int_a \omega^{(\ell)}, \qquad b_\ell \equiv \int_b \omega^{(\ell)}, \tag{A.1}$$

one finds that

$$\begin{aligned} A_j^\ell &= 2a_\ell\omega^{(j-1)\ell} \\ B^{\ell k} &= 2\left[b_\ell\left(\omega^{(k-1)\ell} - \omega^{(N-1)\ell}\right) + a_\ell\frac{\omega^{k\ell}-1}{\omega^\ell - 1}\right]. \end{aligned} \tag{A.2}$$

This implies that

$$(A^{-1})^j_\ell = \frac{1}{a_\ell}\frac{\omega^{(1-j)\ell} - \omega^\ell}{2N}\,, \tag{A.3}$$

so that

$$\begin{aligned}\widetilde{\tau}^{jk} &= (A^{-1})^j_\ell B^{\ell k} = \sum_{\ell=1}^{N-1} \frac{\omega^{-j\ell}-1}{N}\left[\frac{b_\ell}{a_\ell}(\omega^{k\ell}-1) + \frac{\omega^{k\ell}-1}{1-\omega^{-\ell}}\right] \\ &= \Theta^{jk} + \frac{1}{N}\sum_\ell \frac{b_\ell}{a_\ell}(\omega^{-j\ell}-1)(\omega^{k\ell}-1)\,, \end{aligned} \tag{A.4}$$

where

$$\Theta^{jk} = \begin{cases} 1 & \text{for } j \le k\,, \\ 0 & \text{for } j > k\,. \end{cases} \tag{A.5}$$

The second line in (A.4) is obtained using the identity

$$\sum_{\ell \neq N} \frac{\omega^{(a+1)\ell}}{\omega^\ell - 1} = \frac{N-1}{2} - a + N\left[\frac{a}{N}\right], \tag{A.6}$$

where $[x]$ denotes the integer part of x. This identity can be shown by writing the left side as

$$\sum_{\ell\neq N}\left(\frac{\omega^{(a+1)\ell}-1}{\omega^\ell-1} + \frac{1}{\omega^\ell - 1}\right) = \sum_{\ell \neq N}\frac{1}{\omega^\ell-1} + \sum_{\ell\neq N}\sum_{p=0}^{a}\omega^{p\ell}\,, \tag{A.7}$$

and using the identity (B.7) from appendix B.

So we need to expand a_ℓ and b_ℓ about $\widetilde{q} = 0$. More explicitly,

$$\begin{aligned} a_\ell &= \frac{1}{2}\oint_{\widetilde{\alpha}_1} \frac{x^{\ell-1}dx}{\sqrt{(x^N-1)^2 - \widetilde{q}x^{2N}}} \\ b_\ell &= \int_0^{(1+\sqrt{\widetilde{q}})^{-1/N}} \frac{x^{\ell-1}dx}{\sqrt{(x^N-1)^2-\widetilde{q}x^{2N}}}\,. \end{aligned} \tag{A.8}$$

a_ℓ can be evaluated using the residue theorem, with the result:

$$a_\ell = \frac{i\pi}{N}\left[1 + \frac{\ell(N+\ell)}{4N^2}\widetilde{q} + \mathcal{O}(\widetilde{q}^2)\right]. \tag{A.9}$$

Since b_ℓ is singular at $\widetilde{q} = 0$, it must be evaluated by a less direct method. First, we recognize b_ℓ as the hypergeometric function [24]

$$b_\ell = \frac{1}{N(1+\sqrt{\widetilde{q}})^{\ell/N}}\, B\left(\frac{\ell}{N},\frac{1}{2}\right)\cdot F\left[\frac{1}{2},\frac{\ell}{N},\frac{\ell}{N}+\frac{1}{2},\frac{1-\sqrt{\widetilde{q}}}{1+\sqrt{\widetilde{q}}}\right]. \tag{A.10}$$

Expanding about $\widetilde{q}=0$ gives

$$b_\ell = \frac{1}{N}\left(1-\frac{\ell}{N}\sqrt{\widetilde{q}}+\frac{\ell(\ell+N)}{2N^2}\widetilde{q}+\cdots\right)\times$$
$$\times\left\{\left[-2\gamma-\psi\left(\frac{1}{2}\right)-\psi\left(\frac{\ell}{N}\right)-\ln y\right]\left(1+\frac{\ell}{2N}y+\frac{3\ell(\ell+N)}{16N^2}y^2+\cdots\right)-\right.$$
$$\left.-\frac{1}{2}y+\frac{\ell^2-5\ell N-3N^2}{16N^2}y^2+\cdots\right\}, \tag{A.11}$$

where

$$y \equiv \frac{2\sqrt{\widetilde{q}}}{1+\sqrt{\widetilde{q}}}, \tag{A.12}$$

$\gamma = .577215\ldots$ is the Euler constant, and $\psi(x) = d\ln\Gamma(x)/dx$. Then, to order $\widetilde{q}$ we have, using $\gamma+\psi(1/2) = -2\ln 2$,

$$\frac{b_\ell}{a_\ell} = \frac{\ln\widetilde{q}}{2\pi i}+\frac{i}{\pi}\left(\ln 2-\gamma-\psi\left(\frac{\ell}{N}\right)\right)+\frac{i}{\pi}\left(\frac{\ell^2-\ell N-N^2}{4N^2}\right)\widetilde{q}\,. \tag{A.13}$$

Inserting this in (A.4), using Gauss' theorem for ψ of a rational argument,

$$\psi\left(\frac{\ell}{N}\right)+\gamma = -\ln N-\frac{i\pi}{2}\frac{\omega^\ell+1}{\omega^\ell-1}+\frac{1}{2}\sum_{j=1}^{N-1}\left(\omega^{j\ell}+\omega^{-j\ell}\right)\ln\left(\omega^{j/2}-\omega^{-j/2}\right), \tag{A.14}$$

and doing the sum then gives (5.14).

B. Details of the one-instanton contribution to τ^{jk}

The one-instanton prepotential (5.38) can be rewritten as

$$\mathcal{F}_1 = \frac{\pi^{2N}C_1'}{4\pi i}\left(F_1-2^{-2N}F_2\right), \tag{B.1}$$

with

$$F_1 \equiv \sum_{c=1}^{N}\left(\frac{v_c^N}{\Pi_c}\right)^2$$
$$F_2 \equiv \sum_{c,d=1}^{N}\frac{1}{\Pi_c\Pi_d}(v_c+v_d)^{2N} \tag{B.2}$$

and

$$\Pi_c \equiv \prod_{b\neq c}(v_b-v_c)\,. \tag{B.3}$$

τ_1^{jk} is obtained from $\mathcal{F}_1$ by taking second partial derivatives as discussed in (5.36),

$$\tau^{jk} = \frac{1}{2}(\partial_j-\partial_N)(\partial_k-\partial_N)\mathcal{F}_1\,, \tag{B.4}$$

where we have introduced the notation $\partial_a = \partial/\partial v_a$. We therefore need to compute $\partial_a\partial_b F_{1,2}$ at the special vacuum ($v_a = \omega^a$ where $\omega = e^{2\pi i/N}$).

We start with F_1. Taking derivatives of (B.2) and evaluating at the special vacuum gives

$$\partial_a\partial_b F_1 = \sum_c \frac{1}{\Pi_c^2}\Bigg\{ \left(\frac{2N\delta_{ac}}{\omega^c} - \frac{2(1-\delta_{ac})}{\omega^a-\omega^c} + \sum_{d\neq c}\frac{2\delta_{ac}}{\omega^d-\omega^c}\right)\times \\ \times\left(\frac{2N\delta_{bc}}{\omega^c} - \frac{2(1-\delta_{bc})}{\omega^b-\omega^c} + \sum_{d\neq c}\frac{2\delta_{bc}}{\omega^d-\omega^c}\right) - \\ -\frac{2N\delta_{ac}\delta_{bc}}{\omega_c^2} + \frac{2(1-\delta_{ac})(\delta_{ab}-\delta_{cb})}{(\omega^a-\omega^c)^2} - \sum_{d\neq c}\frac{2\delta_{ac}(\delta_{db}-\delta_{cb})}{(\omega^d-\omega^c)^2}\Bigg\}. \tag{B.5}$$

The following lemmas prove to be useful in the evaluation of the above sums:

$$\Pi_c = (-)^{N-1}\frac{N}{\omega^c}, \tag{B.6}$$

$$\sum_{f\neq c}\frac{1}{\omega^f-\omega^c} = \frac{1-N}{2\omega^c} \tag{B.7}$$

and

$$\sum_{f\neq c}\frac{1}{(\omega^f-\omega^c)^2} = \frac{(N-1)(5-N)}{12\omega^{2c}}, \tag{B.8}$$

where (B.6) is evaluated in the special vacuum. (B.6) follows from the fact that $\prod_{a\neq N}(z-\omega^a) = (z^N-1)/(z-1)$ and $\lim_{z\to 1}(z^N-1)/(z-1) = N$. Similarly, (B.7) follows from $\sum_{f\neq N}(1-\omega^f)^{-1} = N^{-1}\lim_{z\to 1}\partial_z[(z^N-1)/(z-1)] = (N-1)/2$. Finally, (B.8) follows from $[\sum_{f\neq N}(1-\omega^f)^{-1}]^2 - \sum_{f\neq N}(1-\omega^f)^{-2} = N^{-1}\lim_{z\to 1}\partial_z^2[(z^N-1)/(z-1)]$. Then, somewhat lengthy algebra gives

$$\partial_a\partial_b F_1 = \frac{1}{N^2}\left[2(2N-1) + \delta_{ab}\frac{1}{3}(N^2-1) + (1-\delta_{ab})\frac{4\omega^a\omega^b}{(\omega^a-\omega^b)^2}\right]. \tag{B.9}$$

The derivatives of F_2 are more complicated:

$$\partial_a\partial_b F_2 = 2\sum_{L=0}^{2N}\binom{2N}{L}\left[f_{2N-L}\partial_a\partial_b f + \partial_a f_{2N-L}\cdot\partial_b f\right], \tag{B.10}$$

where

$$f_L \equiv \sum_c \frac{v_c^L}{\Pi_c}. \tag{B.11}$$

Using our lemmas (B.6)–(B.8) and the identity (A.6) from appendix A, after some algebra one finds in the special vacuum that

$$f_L = (-)^{N-1}\begin{cases} 1 & L = -1 \bmod N \\ 0 & \text{otherwise}, \end{cases} \tag{B.12}$$

$$\partial_a f_L = (-)^{N-1}\omega^{aL}\begin{cases} 0 & 0\le L < N \\ 1 & N \le L < 2N \\ 2 & L = 2N, \end{cases} \tag{B.13}$$

and

$$\partial_a \partial_b f_L = (-)^{N-1} \frac{L-1}{N} \left(1 + \delta_{ab}(L-N)\right) \qquad \text{if } L = 1 \bmod N\,. \tag{B.14}$$

From (B.10) and (B.12) we only need the $L = 1 \bmod N$ case in (B.14). Inserting these into (B.10) gives

$$\partial_a \partial_b F_2 = 2 \binom{2N}{N+1} \left[\frac{2N+1}{N} + \delta_{ab} \right]. \tag{B.15}$$

Finally, substituting (B.9) and (B.15) into (B.4) gives (5.39).

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